M. Math. Ist Year
Midsemestral Examination
Complex Analysis
Instructor — B. Sury
February 19, 2024 - Be BRIEF.

Q 1. Determine $\int_0^{2\pi} e^{e^{i\theta}-i\theta} d\theta$.

OR

Q 1. Recall that the complex line integral with respect to $\overline{dz}$ over a piece smooth path $C : z = \gamma(t)$ is defined as $\int_C f \ \overline{dz} := \overline{\int_C \overline{f} dz}$. If p is a polynomial, evaluate $\int_C p \ \overline{dz}$, where C is the circle $|z - z_0| = r$.

Q 2. Consider the power series $f(z) := \sum_{n\geq 0} c_n z^n$ where $c_0 = 1, c_1 = -1, c_{n+2} = \frac{-4c_{n+1}+c_n}{3}$ for all $n \geq 0$. Write $f(z) = \frac{p(z)}{q(z)}$, where p, q are polynomials. Find the radius of convergence of f.
Hint. Does $q(z) = z^2 - 4z - 3$ work?

OR

Q 2. Consider the power series $f(z) := \sum_{n\geq 1} \frac{z^n}{n^{\log n}}$ and the series $g(z)$ obtained by term-by-term differentiation 2024 times. Prove that the series for g converges absolutely for $|z| \leq 1$.
Hint. For $|z| = 1$, get an upper bound for the n-th coefficient of g as $\frac{1}{n^2}$ for large n onwards.

Q 3. Determine the unique function u which is harmonic on an infinite vertical strip $[s, t] \times (-\infty, \infty)$ and, on the vertical lines $Re(z) = s$ and $Re(z) = t$, it takes constant values 5 and 10 respectively.

OR

Q 3. Let $f \in Hol(D)$, where D is the open unit disc. If $|f(z)| \leq M$ for all $z \in D$, prove that $|f'(z)| \leq \frac{M}{1-|z|}$ for all $z \in D$.
Hint. For $a \in D$, show $|f'(a)| \leq M/(t - |a|)$ for each $|a| < t < 1$.

Q 4. If f is a continuous function on $\mathbb{C}$ which is holomorphic both on the upper half-plane and the lower half-plane, prove that f is entire.
Hint. Show that the integral of f over any rectangular contour R with sides $x = a, y = -b < 0, x = c, y = d > 0$ is zero, as follows. Take R_1 be the rectangle with sides $x = a, x = c, y = d, y = \epsilon > 0$ which is above the x-axis. Consider similarly R_1' below the x-axis whose sides are $x = a, x = c, y = -b, y = -\epsilon < 0$. Compare the sums of the integrals over R_1 and R_1' to the integral over R, as $\epsilon \to 0$. Just quote Morera's theorem then.

OR

Q 4. Show that a non-constant polynomial $p = \sum_{i=0}^{n} c_i z^i$ with complex coefficients has a zero, necessarily using the following idea. If p has no zeroes, observe that the integral $\int_0^{2\pi} \frac{d\theta}{q(\theta+\theta^{-1})}$ does not vanish, where $q(z) = (\sum_i c_i z^i)(\sum_i \overline{c_i}\ z^i)$. Reinterpret this real integral as a contour integral $\int_{|z|=1} \frac{f(z)dz}{g(z)}$ where f, g are polynomials over $\mathbb{C}$ and g has no zeroes. Use Cauchy's theorem to deduce a contradiction to the non-vanishing of q.

Q 5. Prove that there is no conformal map S from the punctured open unit disc $A_{0,1} = \{z : 0 < |z| < 1\}$ ONTO an annulus $A_{1,r} = \{z : 1 < |z| < r\}$ for some $r > 1$.
Hint. For any possible S, show 0 is a removable singularity, and apply the open mapping theorem to the holomorphic function T that is an extension of S to the open unit disc.

OR

Q 5. Find an injective, holomorphic function from the open unit disc D to the set $D \setminus \{[0, 1)\}$.
Hint. Think of composing a map to the lower half-plane $\{z : Im(z) < 0\}$ and a map from the latter to $D \setminus \{[0, 1)\}$.

Q 6. Let $f : D \to \mathbb{C}$ be a function such that f^2, f^3 are in $Hol(D)$, where D is the open unit disc. Prove that $f \in Hol(D)$.
Hint. If z_0 is a zero of f, write $f^2 = (z - z_0)^r g$, $f^3 = (z - z_0)^s h$ where g, h do not vanish at z_0. What is the relation between r and s?

OR

Q 6. Consider the "truncated exponential polynomial" $E_n(z) = \sum_{k=0}^{n} \frac{z^k}{k!}$ for any $n \geq 2$. If $z_1, \cdots, z_n$ are the roots of E_n, show that $\sum_{i=1}^{n} 1/z_i^2 = 0$. State (no need to prove) what other powers z_i^{-d} can be taken instead of $1/z_i^2$ above.
Hint. Show first that z_i's must be distinct. Then, apply Cauchy's theorem for $z^{n-2}/E_n(z)$ on a sufficiently large circle.